Combining Probabilistic Estimates to Reduce Uncertainty

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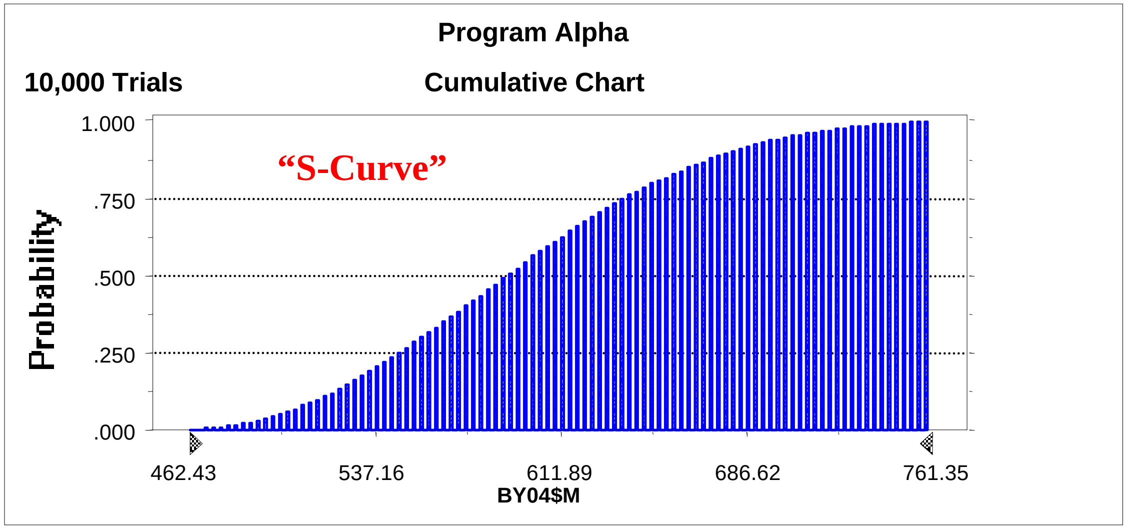
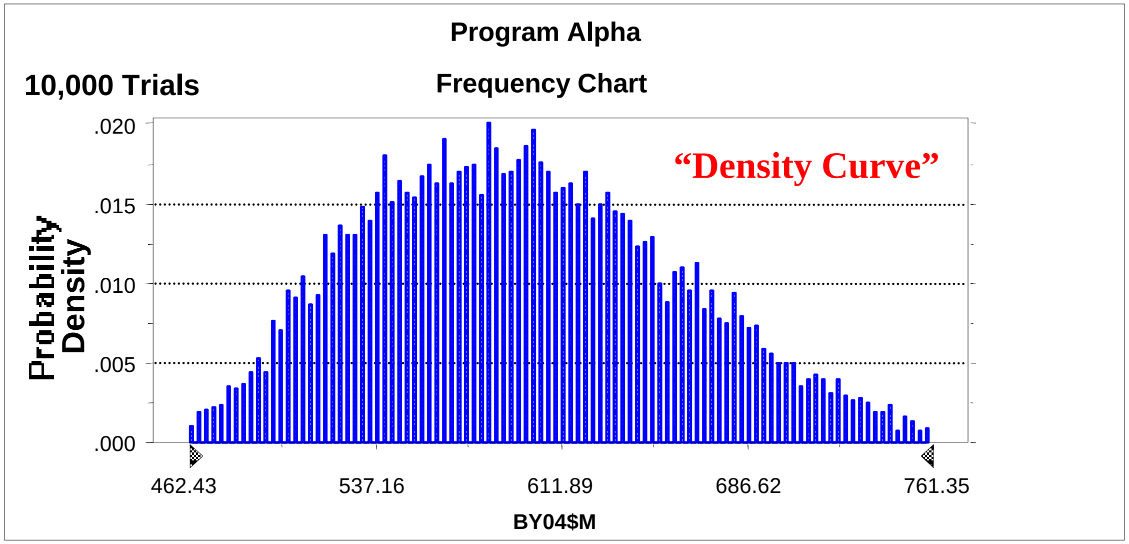
What a Cost Estimate Looks Like

Percentile	Value
10%	516.81
20%	538.98
30%	557.85
40%	575.48
50%	592.72
60%	609.70
70%	629.19
80%	650.97
90%	683.01

Statistics	Value
Trials	10,000
Mean	596.40
Median	592.72
Mode	---
Standard Deviation	63.18
Range Minimum	450.19
Range Maximum	796.68





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Objective of the Study

- **We Have *n* Estimates, Independent of Each Other, of the Same System or Project**
- **The Estimates are Expressed Statistically – Their S-Curves, Means, and Standard Deviations are Known**
- **We Want to Combine These *n* Estimates to Obtain One Estimate that Contains Less Uncertainty than Each of the *n* Estimates Individually**
- **Question: Will the Combined Estimate Actually be Less Uncertain than Each of the *n* Independent Estimates Individually**

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Assumptions

- **The *n* Estimates are Independent of Each Other**
- **We Know Each of their Means, Standard Deviations, and therefore their Coefficients of Variation (= Standard Deviation ÷ Mean)**
- **The Estimates are "Credible," i.e. …**
  - **They are Based on the Same Technical Description of the Program and on Risk Assessments Validly Drawn from the Same Risk Information Available to Each Estimating Team**
  - **Each Estimating Team Applied Appropriate Mathematical Techniques to Obtain the Estimate and Conduct the Cost-Risk Analysis, Including (for example) Inter-Element Correlations when Appropriate**
  - **Each Estimating Team Worked from the Same Ground Rules, but May Have Applied Different Estimating Methods and Made Different Assumptions when Encountering the Absence of Some Required Information**

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A Note on “Credibility”

- A Cost Estimate is “Credible” if it is Based on …
  - A Valid Technical and Programmatic Description of the Item or Program to be Costed
  - Application of Valid Mathematical Techniques that Produce Cost Estimates from Technical and Programmatic Information
  - An Assessment of Program Risks by Knowledgeable Engineers and their Translation into Cost Impacts by Knowledgeable Cost Analysts
  - Application of Valid Statistical Methods in “Rolling Up” WBS-Cost Probability Distributions into a Total-Cost Probability Distribution
- Whether or Not an Estimate is Credible does not Depend on its Accuracy or Precision
- All Estimates Remaining after the Reconciliation Process Has Been Completed Should be Considered Credible

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Mathematical Framework

- **Denote, for $1 \leq k \leq n$, the k^{th} Estimate as the Random Variable S_k**
- **Denote, for $1 \leq k \leq n$, the Mean ("Expected Value") of the k^{th} Estimate as the Number $\mu_k = E(S_k)$**
- **Denote, for $1 \leq k \leq n$, the Standard Deviation ("Sigma Value") of the k^{th} Estimate as the Number $\sigma_k = \sqrt{Var(S_k)}$**
- **Denote, for $1 \leq k \leq n$, the Coefficient of Variation of the k^{th} Estimate as the Number $\theta_k = \sigma_k / \mu_k$**
  - **The Coefficient of Variation is the Expression of the Standard Deviation as a Percentage of the Mean – a Common Measure of the Uncertainty of a Random Variable**
  - **The Coefficient of Variation is a Measure of the Precision of the Estimate**

Note: "$Var(S_k)$" is the Variance of the Random Variable, which is the Square of the Standard Deviation.

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A Basic Mathematical Fact

Theorem: If, for $1 \le k \le n$, the numbers μ_k are all positive, then

$$\sum_{k=1}^{n} \mu_k^2 < \left(\sum_{k=1}^{n} \mu_k \right)^2 .$$

Proof:

$$\left(\sum_{k=1}^{n} \mu_k \right)^2 = \left(\mu_1 + \mu_2 + \mu_3 + \dots \right)^2$$

$$= \mu_1^2 + \mu_2^2 + \mu_3^2 + \dots + 2\mu_1\mu_2 + 2\mu_1\mu_3 + 2\mu_2\mu_3 + \dots$$

$$> \mu_1^2 + \mu_2^2 + \mu_3^2 + \dots = \sum_{k=1}^{n} \mu_k^2$$

QED

Recall from Algebra: $(a+b)^2 = a^2 + b^2 + 2ab$.

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Example: The Case of Three Independent Estimates

- **Suppose We Have Three Independent (of each other) Probabilistic Estimates of the Same System or Project: S_1, S_2, S_3**
- **The Statistical Average (Mean) of the Three Estimates is**

$$\bar{S} = \frac{S_1 + S_2 + S_3}{3}$$

- **Because the Mean of a Sum is the Sum of the Means (a statistical theorem), We Know that the Mean of the Average Estimate is**

$$\mu = E\left(\bar{S}\right) = \frac{\mu_1 + \mu_2 + \mu_3}{3}$$

- **Furthermore, the Variance of a Sum of <u>Independent</u> Random Variables is the Sum of the Individual Variances with Any Constant Divisor or Multiple Squared, We See that**

$$\sigma^2 = Var\left(\bar{S}\right) = \frac{Var\left(S_1\right) + Var\left(S_2\right) + Var\left(S_3\right)}{9} = \frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{9}$$

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The Role of the Coefficients of Variation

- Suppose $\theta_1 = \sigma_1/\mu_1$, $\theta_2 = \sigma_2/\mu_2$, and $\theta_3 = \sigma_3/\mu_3$ are the Respective Coefficients of Variation – the Three Standard Deviations Expressed as Percentages of their Corresponding Means
- Then the Standard Deviation of the Average Estimate Can be Expressed as

$$\sigma = \sqrt{\frac{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{9}} = \frac{1}{3}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{1}{3}\sqrt{\theta_1^2\mu_1^2 + \theta_2^2\mu_2^2 + \theta_3^2\mu_3^2}$$

- Now Suppose We Denote by θ the Coefficient of Variation of the Average of the Three Estimates – Then

$$\theta = \frac{\sigma}{\mu} = \frac{\frac{1}{3}\sqrt{\theta_1^2\mu_1^2 + \theta_2^2\mu_2^2 + \theta_3^2\mu_3^2}}{\frac{1}{3}(\mu_1 + \mu_2 + \mu_3)} = \frac{\sqrt{\theta_1^2\mu_1^2 + \theta_2^2\mu_2^2 + \theta_3^2\mu_3^2}}{(\mu_1 + \mu_2 + \mu_3)}$$

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The Relative Uncertainty of the Average Estimate

- Use the Symbol λ to Denote the Ratio of the Uncertainty (as expressed by the coefficient of variation) of the Average Estimate to the Uncertainty of the "Best" (smallest uncertainty) Original Estimate
- Algebraically, this Means that

$$\lambda = \frac{\theta}{min(\theta_1, \theta_2, \theta_3)} = \frac{\sqrt{\theta_1^2 \mu_1^2 + \theta_2^2 \mu_2^2 + \theta_3^2 \mu_3^2}}{(\mu_1 + \mu_2 + \mu_3) \times min(\theta_1, \theta_2, \theta_3)}$$

- Therefore the Uncertainty of the Average Estimate is $\lambda \times min(\theta_1, \theta_2, \theta_3)$

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A Numerical Example with Equal Coefficients of Variation

- Suppose We Have Three Independent Probabilistic Estimates of a System or Project
- Suppose the Three Means are $\mu_1 = 400$, $\mu_2 = 500$, and $\mu_3 = 600$ Million Dollars, Respectively
- Suppose the Three Coefficients of Variation are All 20%, i.e., $\theta_1 = 0.20$, $\theta_2 = 0.20$, and $\theta_3 = 0.20$, Respectively
- This Implies that the Three Standard Deviations are, Respectively, $\sigma_1 = 80$, $\sigma_2 = 100$, and $\sigma_3 = 120$
- Then the Average Estimate is $\mu = (400+500+600)/3 = 500$, and its Standard Deviation is

$$\sigma = \frac{1}{3}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{1}{3}\sqrt{6400 + 10000 + 14400} = \frac{1}{3}\sqrt{30800} = 58.5$$

- This Implies that the Coefficient of Variation of the Average Estimate is $\theta = \sigma/\mu = 58.5/500 = 0.117 = 11.7\%$, so that the Average Estimate Has Less Uncertainty than Each of the Three Original Estimates

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An Alternative Calculation

- We Can Make the Same Calculation by Going Directly to the Expression for λ
- The Ratio of the Uncertainty (as expressed by the coefficient of variation) of the Average Estimate to the Uncertainty of the "Best" (smallest uncertainty) Original Estimate is

$$\lambda = \frac{\theta}{\min(\theta_1, \theta_2, \theta_3)} = \frac{\sqrt{\theta_1^2\mu_1^2 + \theta_2^2\mu_2^2 + \theta_3^2\mu_3^2}}{(\mu_1 + \mu_2 + \mu_3) \times \min(\theta_1, \theta_2, \theta_3)}$$

$$= \frac{\sqrt{(0.20^2)(400^2) + (0.20^2)(500^2) + (0.20^2)(500^2)}}{(400 + 500 + 600) \times \min(0.20, 0.20, 0.20)}$$

$$= \frac{\sqrt{(0.04)(160000) + (0.04)(250000) + (0.04)(360000)}}{(1500) \times 0.20}$$

$$= \frac{\sqrt{6400 + 10000 + 14400}}{300} = \frac{\sqrt{30800}}{300} = 0.585$$

- Therefore the Resulting Uncertainty of the Average Estimate is $\lambda \times \min(\theta_1, \theta_2, \theta_3)$ = 0.585×0.20 = 0.117 = 11.7%

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What Do We Conclude About Our Numerical Example?

- We Began with Three Independent (from each other) Probabilistic Estimates, the Standard Deviation of Each of which was 20% of its Respective Mean
- Upon Averaging the Means of Each of the Three Estimates, We Obtained an Estimate whose Standard Deviation was Only 11.7% of the Mean
- In this Case, therefore, We Were Able to Use the Three Estimates to Derive One Estimate that Has Less Uncertainty than Each of the Three Original Estimates

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An Excursion

- **Suppose We Have Three Probabilistic Estimates of a System or Project as Before, whose Three Means are $\mu_1 = 400$, $\mu_2 = 500$, and $\mu_3 = 600$ Million Dollars, Respectively, as Before**
- **However, Suppose the Three Coefficients of Variation are Now, Respectively, $\theta_1 = 10\%$, $\theta_2 = 30\%$, and $\theta_3 = 50\%$**
- **This Implies that the Three Standard Deviations are, Respectively, $\sigma_1 = 40$, $\sigma_2 = 150$, and $\sigma_3 = 300$**
- **The Average Estimate is Again $\mu = (400+500+600)/3 = 500$, but its Standard Deviation is Now**

$$\sigma = \frac{1}{3}\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{1}{3}\sqrt{1600 + 22500 + 90000} = \frac{1}{3}\sqrt{114100} = 112.6$$

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Naïve Analysis of the Excursion

- **All This Implies that the Coefficient of Variation of the Average Estimate is $\theta = \sigma/\mu$ = 112.6/500 = 0.225 = 22.5%, so that the Average Estimate Has Less Uncertainty than Two of the Three Original Estimates, but <u>not</u> the Third**
- **In this Case, <u>It Appears</u> we would be Better Off (at least with respect to uncertainty) by Using the First Estimate (which has a 10% coefficient of variation) than to Average the Three Estimates**

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Fixing the Excursion by Weighting the Estimates

- Because the Three Estimates in the Excursion are of Different Levels of Precision, to Combine Them We Should Really be Calculating their "Weighted Average" (weighted by precision), rather than their Straight Average
- Again, Suppose the Three Means are $\mu_1 = 400$, $\mu_2 = 500$, and $\mu_3 = 600$ Million Dollars, Respectively, as Before, and the Three Coefficients of Variation are, Respectively, $\theta_1 = 10\%$, $\theta_2 = 30\%$, and $\theta_3 = 50\%$, again as Before
- This again Implies that the Three Standard Deviations are, Respectively, $\sigma_1 = 40$, $\sigma_2 = 150$, and $\sigma_3 = 300$
- Suppose Now We Calculate the Weighted Average Estimate (denoted by $\overline{W}$), Using the Coefficients of Variation as the Respective Weights – in Particular,

$$\overline{W} = \frac{\dfrac{\mu_1}{\theta_1} + \dfrac{\mu_2}{\theta_2} + \dfrac{\mu_3}{\theta_3}}{\dfrac{1}{\theta_1} + \dfrac{1}{\theta_2} + \dfrac{1}{\theta_3}}$$

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Calculating the Weighted Average

- **The Formula for the Weighted Average Counts Less Uncertain Estimates, i.e. those Having a Smaller Coefficient of Variation, More Heavily in the Average**
- **Let's Now Calculate the Weighted Average Estimate:**

$$\overline{W} = \frac{\frac{\mu_1}{\theta_1}+\frac{\mu_2}{\theta_2}+\frac{\mu_3}{\theta_3}}{\frac{1}{\theta_1}+\frac{1}{\theta_2}+\frac{1}{\theta_3}} = \frac{\frac{400}{0.10}+\frac{500}{0.30}+\frac{600}{0.50}}{\frac{1}{0.10}+\frac{1}{0.30}+\frac{1}{0.50}} = \frac{4000+1666.66667+1200}{10+3.33333+2}$$

$$= \frac{6866.66667}{15.33333} = 447.82618$$

- **Note that the Weighted Average is Closer to 400 than is the "Unweighted" Average, because the 400 is Weighted More Heavily than Either the 500 or 600**

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Variance of the Weighted Average

- **The Weighted Average is Really the Expected Value of the Weighted Probabilistic Estimate, which is**

$$W = \frac{\frac{S_1}{\theta_1} + \frac{S_2}{\theta_2} + \frac{S_3}{\theta_3}}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}}$$

- **The Variance of the Weighted Estimate then has the formula**

$$Var(W) = \frac{Var\left(\frac{S_1}{\theta_1} + \frac{S_2}{\theta_2} + \frac{S_3}{\theta_3}\right)}{\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}\right)^2} = \frac{\frac{1}{\theta_1^2} Var(S_1) + \frac{1}{\theta_2^2} Var(S_2) + \frac{1}{\theta_3^2} Var(S_3)}{\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}\right)^2}$$

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Standard Deviation of the Weighted Average

- Because the Three Means are $\mu_1 = 400$, $\mu_2 = 500$, and $\mu_3 = 600$ Million Dollars, Respectively, and the Three Coefficients of Variation are $\theta_1 = 10\%$, $\theta_2 = 30\%$, and $\theta_3 = 50\%$, Respectively, We See that the Three Standard Deviations are $\sigma_1 = 40$, $\sigma_2 = 150$, and $\sigma_3 = 300$, Respectively
- Therefore the Variance of the Weighted Average is

$$Var(W) = \frac{\frac{1}{\theta_1^2}Var(S_1) + \frac{1}{\theta_2^2}Var(S_2) + \frac{1}{\theta_3^2}Var(S_3)}{\left(\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}\right)^2} = \frac{\frac{40^2}{(0.10)^2} + \frac{150^2}{(0.30)^2} + \frac{300^2}{(0.50)^2}}{\left(\frac{1}{0.10} + \frac{1}{0.30} + \frac{1}{0.50}\right)^2}$$

$$= \frac{160{,}000 + 250{,}000 + 360{,}000}{(10 + 3.33333 + 2)^2} = \frac{770{,}000}{(15.33333)^2} = 3{,}275.04868$$

- It Follows that the Standard Deviation of the Weighted Average of the Three Estimates is the Square Root of $3{,}275.05$, namely 57.23

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Coefficient of Variation of the Weighted Average

- **On Previous Slides, We Found that $\overline{W} = 447.83$ and $\sigma(W) = 57.23$**
- **It therefore Follows that the Coefficient of Variation of the Weighted Average is**

$$\theta_W = \frac{\sigma(W)}{\overline{W}} = \frac{57.22804}{447.82618} = 0.12779 = 12.78\%$$

- **The Lesson Here Seems to be that, if We Use Coefficients of Variation to <u>Weight these Particular</u> Statistically Independent Probabilistic Estimates**
  - **… the Weighted Estimate has Significantly Less Uncertainty than the Unweighted "Average" Estimate (whose coefficient of variation was 22.5%)**
  - **… and Less Uncertainty than Two of the Three Original Estimates**
  - **… but Still not Less Uncertainty than the Least Uncertain of the Three Original Estimates**

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The General Case

- **Suppose We Have Three Independent Probabilistic Estimates of a System or Project**
  - **Suppose the Three Means are μ_1, μ_2, and μ_3**
  - **Suppose the Three Coefficients of Variation are, Respectively, θ_1, θ_2, and θ_3**
  - **This Implies that the Three Standard Deviations are, Respectively, $\sigma_1 = \theta_1\mu_1$, $\sigma_2 = \theta_2\mu_2$, and $\sigma_3 = \theta_3\mu_3$**
- **We Can Assume, without Loss of Generality, that $\theta_1 \le \theta_2 \le \theta_3$**
  - **Furthermore, We Can Set $\theta_2 = \alpha\theta_1$ and $\theta_3 = \beta\theta_1$**
  - **It Therefore Follows that $1 \le \alpha \le \beta$**
- **This Leads to $\sigma_1 = \theta_1\mu_1$, $\sigma_2 = \alpha\theta_1\mu_2$, and $\sigma_3 = \beta\theta_1\mu_3$**

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Weighting the Estimates in the General Case

- **Suppose Now We Calculate the Weighted Average Estimate (denoted by $\overline{W}$), Using the Coefficients of Variation as the Respective Weights – in Particular,**

$$\overline{W} = \frac{\frac{\mu_1}{\theta_1} + \frac{\mu_2}{\theta_2} + \frac{\mu_3}{\theta_3}}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} = \frac{\frac{\mu_1}{\theta_1} + \frac{\mu_2}{\alpha\theta_1} + \frac{\mu_3}{\beta\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\alpha\theta_1} + \frac{1}{\beta\theta_1}}$$

$$= \frac{\frac{1}{\theta_1}\left(\mu_1 + \frac{\mu_2}{\alpha} + \frac{\mu_3}{\beta}\right)}{\frac{1}{\theta_1}\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)} = \frac{\mu_1 + \frac{\mu_2}{\alpha} + \frac{\mu_3}{\beta}}{1 + \frac{1}{\alpha} + \frac{1}{\beta}}$$

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Calculating the Variance of the Weighted Average Estimate

- **As Before, the Weighted Average Estimate is Really the Expected Value of the Weighted Probabilistic Estimate, which is**

$$W = \frac{\frac{S_1}{\theta_1} + \frac{S_2}{\theta_2} + \frac{S_3}{\theta_3}}{\frac{1}{\theta_1} + \frac{1}{\theta_2} + \frac{1}{\theta_3}} = \frac{\frac{S_1}{\theta_1} + \frac{S_2}{\alpha\theta_1} + \frac{S_3}{\beta\theta_1}}{\frac{1}{\theta_1} + \frac{1}{\alpha\theta_1} + \frac{1}{\beta\theta_1}} = \frac{S_1 + \frac{S_2}{\alpha} + \frac{S_3}{\beta}}{1 + \frac{1}{\alpha} + \frac{1}{\beta}}$$

- **In the Current Situation, the Variance of the Weighted Average Estimate is then (since the estimates are assumed to be independent)**

$$Var(W) = \frac{Var\left(S_1 + \frac{S_2}{\alpha} + \frac{S_3}{\beta}\right)}{\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2} = \frac{Var(S_1) + \frac{1}{\alpha^2}Var(S_2) + \frac{1}{\beta^2}Var(S_3)}{\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2}$$

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The Variance of the Weighted Average Estimate

- **Continuing the Calculation on the Previous Chart and Recalling that $\sigma_1 = \theta_1\mu_1$, $\sigma_2 = \alpha\theta_1\mu_2$, and $\sigma_3 = \beta\theta_1\mu_3$, We See that**

$$Var(W) = \frac{Var(S_1) + \frac{1}{\alpha^2}Var(S_2) + \frac{1}{\beta^2}Var(S_3)}{\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2}$$

$$= \frac{(\theta_1\mu_1)^2 + \frac{1}{\alpha^2}(\alpha\theta_1\mu_2)^2 + \frac{1}{\beta^2}(\beta\theta_1\mu_3)^2}{\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2} = \frac{\theta_1^2\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right)}{\left(1 + \frac{1}{\alpha} + \frac{1}{\beta}\right)^2}$$

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Coefficient of Variation of the Weighted Average Estimate

- **It Follows from the Aforementioned Calculations that the Coefficient of Variation of the Weighted Average Estimate is**

$$\theta_W = \frac{\sigma(W)}{\overline{W}} = \frac{\sqrt{Var(W)}}{\overline{W}} = \frac{\sqrt{\dfrac{\theta_1^2\left(\mu_1^2+\mu_2^2+\mu_3^2\right)}{\left(1+\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)^2}}}{\dfrac{\mu_1+\dfrac{\mu_2}{\alpha}+\dfrac{\mu_3}{\beta}}{1+\dfrac{1}{\alpha}+\dfrac{1}{\beta}}} = \frac{\theta_1\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}}{\mu_1+\dfrac{\mu_2}{\alpha}+\dfrac{\mu_3}{\beta}}$$

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When is the Weighted Average Estimate a “Better” Estimate?

- **According to Our Criterion, We Will Consider the Weighted Average Estimate to be a “Better” Estimate than the Best of the Three Original Estimates if it is Less Uncertain**
- **This Will Happen if $\theta_W < \theta_1$, namely if**

$$\frac{\theta_1\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}}{\mu_1+\frac{\mu_2}{\alpha}+\frac{\mu_3}{\beta}} < \theta_1$$

According to the Calculation on the Previous Chart

- **And so, if** $\sqrt{\mu_1^2+\mu_2^2+\mu_3^2} < \mu_1+\frac{\mu_2}{\alpha}+\frac{\mu_3}{\beta}$
- **… or if, Equivalently,** $\sqrt{\mu_1^2+\mu_2^2+\mu_3^2} - \mu_1 - \frac{\mu_2}{\alpha} - \frac{\mu_3}{\beta} < 0$

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Applying the General-Case Test to the First Numerical Example

- **Our Test Will Assert that the Weighted Average Estimate is a "Better" Estimate than the Best of the Three Original Estimates if**

$$\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}-\mu_1-\frac{\mu_2}{\alpha}-\frac{\mu_3}{\beta}\le 0$$

- **In Our First Numerical Example, $\mu_1 = 400$, $\mu_2 = 500$, and $\mu_3 = 600$ and $\theta_1 = 20\%$, $\theta_2 = 20\%$, and $\theta_3 = 20\%$**
- **It Follows that $\alpha = 1$ and $\beta = 1$, so that**

$$\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}-\mu_1-\frac{\mu_2}{\alpha}-\frac{\mu_3}{\beta}=\sqrt{400^2+500^2+600^2}-400-\frac{500}{1}-\frac{600}{1}$$

$$=\sqrt{770{,}000}-400-500-600$$

$$=877.49644-1{,}500=-622.50536$$

- **Since the Answer is a Negative Number, the Test's Advice is to Use the Weighted Average Estimate as the Cost Estimate**

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Applying the General-Case Test to the Excursion Example

- **Our Test Will Assert that the Weighted Average Estimate is a "Better" Estimate than the Best of the Three Original Estimates if**

$$\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}-\mu_1-\frac{\mu_2}{\alpha}-\frac{\mu_3}{\beta}\le 0$$

- **In Our Excursion Example, μ_1 = 400, μ_2 = 500, and μ_3 = 600 and θ_1 = 10%, θ_2 = 30%, and θ_3 = 50%**
- **It Follows that α = 3 and β = 5, so that**

$$\sqrt{\mu_1^2+\mu_2^2+\mu_3^2}-\mu_1-\frac{\mu_2}{\alpha}-\frac{\mu_3}{\beta}=\sqrt{400^2+500^2+600^2}-400-\frac{500}{3}-\frac{600}{5}$$

$$=\sqrt{770{,}000}-400-166.66667-120$$

$$=877.49644-686.66667=190.82977$$

- **Since the Answer is a Positive Number, the Test's Advice is to Stick with the Best of the Three Original Estimates**

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Summary

- **Suppose We Have Three Independent Probabilistic Estimates of a System or Project**
  - **Suppose the Three Means are μ_1, μ_2, and μ_3**
  - **Suppose the Three Coefficients of Variation are, Respectively, θ_1, θ_2, and θ_3, where $\theta_1 \le \theta_2 \le \theta_3$,**
  - **… so that the Three Standard Deviations are, Respectively, $\sigma_1 = \theta_1\mu_1$, $\theta_2 = \alpha\theta_1\mu_2$, and $\sigma_3 = \beta\theta_1\mu_3$, where $1 \le \alpha \le \beta$**
- **Our Test Recommends that the Weighted Average Estimate be Used if**

$$\sqrt{\mu_1^2 + \mu_2^2 + \mu_3^2} - \mu_1 - \frac{\mu_2}{\alpha} - \frac{\mu_3}{\beta} < 0$$

and the Best of the Three Original Estimates be Used Otherwise

- **The Formulas Easily Generalize to Cover the Case of More (or Fewer) than Three Independent Estimates**

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Conclusion

- If Reducing Uncertainty in Your Estimate is Your Goal, and You Have Several Valid Independent Estimates to Choose from,
  – Is it Better to Average Multiple Estimates or
  – Is it Better Simply to Use the "Best" of Them?
- If the Several Estimates Vary in Uncertainty, We Have Established a Numerical Test to Determine Whether the Weighted (by their respective coefficients of variation) Average Estimate Has Less Uncertainty than the Least Uncertain of the Original Estimates

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